

PAR-LPTHE 02-02

LPTM-cergy

Janvier 2002

 $C_{N+1}^{(2)}$ Ruijsenaars-Schneider models

by

J. AVAN¹ and G. ROLLET²**ABSTRACT**

We define the notion of $C_{N+1}^{(2)}$ Ruijsenaars-Schneider models and construct their Lax formulation. They are obtained by a particular folding of the A_{2N+1} systems. Their commuting Hamiltonians are linear combinations of Koornwinder-van Diejen “external fields” Ruijsenaars-Schneider models, for specific values of the exponential one-body couplings but with the most general 2 double-poles structure as opposed to the formerly studied BC_N case. Extensions to the elliptic potentials are briefly discussed.

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1 Introduction

We wish to present here an explicit Lax formulation for a subclass of one-body extensions of the classical integrable Ruijsenaars-Schneider (RS) models [1] quantized in [2]. The quantum version of these extensions was formulated initially by van Diejen [3, 4, 5] starting from the pioneering works of Koornwinder on algebraic formulations of extensions of MacDonald polynomials [6]. This formulation, and the proof of quantum integrability relied upon analytical arguments using the newly constructed Koornwinder polynomials as a basis for the wave function Hilbert space [4]. A quantum formulation for the elliptic hamiltonians, conjectured in [5], was given in a series of works by Komori et al. [7] using a corner-transfer matrix method combining functional representations of both bulk quantum R -matrix and boundary reflection K -matrices. General one-body extensions of the difference RS operators were thus built, limits of which could be identified with the one-body extensions of differential Calogero-Moser (CM) Hamiltonians constructed by Inozemtsev [8] and associated to BC_N lattices [9].

It must be noticed at this point that no clear classical limit of this construction exists, entailing as it does a coordinate-permutation operator in the functional representation of the quantum bulk R -matrix and a coordinate-reflection operator in the quantum reflection K -matrix. In fact, no full classical Lax formulation exists for the most general Koornwinder-van Diejen (KvD) Hamiltonians; a first step in this direction was our identification of the BC_N Ruijsenaars-Schneider Hamiltonians as linear combinations of particular KvD hamiltonians [10]. In addition the classical r -matrix was obtained in this case, providing an interesting example of a dynamical dependance in *both* sets of dynamical variables, rapidities and positions.

Our purpose is to describe a classical Lax formulation of a more general subclass of KvD potentials; more precisely the one-body part of the potential will exhibit the same 2 double-pole dependence as the full KvD; the coupling constants however, identified with the residues, still depend on one single coupling. This however marks

a qualitative advance compared to the previous construction of BC_N RS model, since one there had only one double and *one single* pole.

As in the previous case we will rely on a consistent Z_2 folding procedure applied to a suitable A_n RS model to obtain new classical integrable systems and their Lax formulation. The first suggestion of constructing (in that case) BC_N and C_N RS models by Z_2 folding of $A_{2N(+1)}$ RS model came from Ruijsenaars himself [11] and was later explicited at the level of the Lax formulation [12, 13]. A related programme was applied to the simpler case of CM models [9], using folding procedures on the phase-space variables and twistings of the underlying Lie algebras to obtain general one-body extensions (see also [8]).

We here establish that this more general class of KvD potentials [3] (restricted to the hyperbolic case for the sake of simplicity) is obtained as linear combinations of pure RS Hamiltonians which may be described in terms of the root lattice of the twisted affine superalgebra $OSp(2|2N)^{tw}$ or $C_{N+1}^{(2)}$. The lattice characterizing the form of the potential indeed exhibits the shift of shortest roots by $\frac{1}{2}$ the derivation, characteristic of the $C_{N+1}^{(2)}$ root lattice [14]³. At this time however we lack a deeper interpretation of the occurrence of this particular lattice.

The problem of finding the Lax formulation for the most general four-coupling hyperbolic KvD potentials remains open at this point, but we conjecture that this subclass, exhibiting as it does the full pole structure, constitutes the best starting point for eventual achievement of this construction.

The detailed analysis of elliptic generalizations will be left for later studies. However it is established here that the most general 8 pole structure found in [7] (degenerating to a 4 double-pole structure in the classical limit) may be similarly obtained by a generalized folding operation and the previous conjecture on the hyperbolic case is therefore extendable to the elliptic case.

A final comment regarding the denomination “ $C_{N+1}^{(2)}$ RS model” used here. The

³We are indebted to Luc Frappat for providing this reference and the identification of $C_{N+1}^{(2)}$

qualitative difference between the class of potentials obtained from the initial BC_N folding and this $C_{N+1}^{(2)}$ folding, i.e. the occurrence of a 2 double pole structure in the one-body term instead of a double-pole times single-pole structure, vanishes in the CM limit where poles *add* instead of *multiply*. It follows that the distinction between BC_N and $C_{N+1}^{(2)}$ CM models is essentially non-existent at least at the level of Hamiltonians, and the litterature (see for instance [9]) rightly characterizes only “ BC_N type” models.

2 The $C_{N+1}^{(2)}$ Ruijsenaars-Schneider model

2.1 Invariant subspaces of the A_n RS dynamics

Presentation of the $C_{N+1}^{(2)}$ RS model is easily done in a more general framework where one constructs subsets of the phase space for an A_n RS hierarchy so that they be invariant under a subclass of the flows generated by some specific combinations of the Hamiltonians. We now recall the definition of the A_n RS hierarchy, for any n .

The canonical variables are a set of rapidities $\{\theta_i, i = 1 \cdots n+1\}$ and conjugate positions q_i such that $\{\theta_i, q_j\} = \delta_{ij}$. The Hamiltonians are initially defined as:

$$h_l = \sum_{I \subset \{1 \cdots n+1\}, |I|=l} e^{-\beta \theta_I} f_I \quad \text{where} \quad f_I = \prod_{i \in I, k \notin I} f(q_i - q_k)^{1/2} \quad \text{and} \quad \theta_I = \sum_{i \in I} \theta_i.$$

Function f may take different forms (rational, hyperbolic, trigonometric), resp:

$$f(q) = 1 - \frac{g^2}{q^2} ; \quad f(q) = 1 - \frac{\sinh^2 \gamma}{\sinh^2 \frac{\nu q}{2}} ; \quad f(q) = 1 - \frac{\sin^2 \gamma}{\sin^2 \frac{\nu q}{2}}$$

The most general elliptic case where $f(q) = (\lambda + \nu \mathcal{P}(q))$, \mathcal{P} being Weierstrass function will not be fully explicited here. Our construction may however be extended to it with due modifications to be discussed in the appropriate place (see Remark 2 in this section and Comment 2 in Section 5).

The trigonometric and hyperbolic cases define the same model at least locally up to a redefinition of the parameters (the global structure of trigonometric *vs.* hyperbolic models is however quite different, owing to qualitatively distinct topological

properties, as can be seen for instance in [15]). The rational case is obtained by an easy limit procedure from one of these models. We shall therefore consider in the following only the hyperbolic model.

Let us note that $f(q) = v(q)v(-q)$, with : $v(q) = \frac{\sinh(\frac{\nu q}{2} + \gamma)}{\sinh \frac{\nu q}{2}} = \lambda^{-1/2} \frac{z - \lambda}{z - 1}$. These functions are thus rational functions of an exponential variable, $z = e^{\nu q}$, defining $\lambda = e^{-2\gamma}$. The square root is defined here with a cut on the real negative axis.

Both functions f and v are periodic with period $T \equiv \frac{2i\pi}{\nu}$. One introduces now a better adapted (although possibly degenerated) set of Hamiltonians as:

$$K_0 = \frac{h_{n+1} + h_{n+1}^{-1}}{2} ; \text{ for } l = 1 \cdots \left[\frac{n+1}{2} \right], K_l = h_l, K_{-l} = \frac{h_{n+1-l}}{h_{n+1}} = \sum_{|I|=l} e^{\beta\theta_I} f_I \quad (1)$$

The negative-index Hamiltonians are in fact obtained from traces of *negative powers* of the RS Lax matrix, using its remarkable Cauchy structure [1] to connect L^{-1} with L^t . One then considers any idempotent bijection σ over the index set $\{1 \cdots n+1\}$, separating them into invariant singlets $\sigma(i) = i$ and doublets $\sigma(i) = j, \sigma(j) = i, j \neq i$. It is easily shown that:

Proposition 1 *For any idempotent bijection σ , the manifolds defined by:*

$$\forall i, q_i + q_{\sigma(i)} = 0(T) ; \theta_i + \theta_{\sigma(i)} = 0\left(\frac{2i\pi}{\beta}\right) \quad (2)$$

are kept invariant by the evolution generated by the Hamiltonians $\frac{K_l + K_{-l}}{2} \equiv \mathcal{H}_l$.

Proof: The evolution equations are: $\{q_j, \mathcal{H}_l\} = \beta \sum_{I \ni j} \frac{e^{\beta\theta_I} - e^{-\beta\theta_I}}{2} f_I$ and $\{\theta_j, \mathcal{H}_l\} =$

$$\sum_{I \ni j} \frac{e^{\beta\theta_I} + e^{-\beta\theta_I}}{2} f_I \sum_{k \notin I} (\ln f^{1/2})'(q_j - q_k) - \sum_{I \not\ni j} \frac{e^{\beta\theta_I} + e^{-\beta\theta_I}}{2} f_I \sum_{i \in I} (\ln f^{1/2})'(q_i - q_j). \quad (3)$$

Invariance of the manifolds (2) under (3) is straightforwardly obtained from the parity and T -periodicity of the two-body potential function f ; the explicit θ parity and $\frac{2i\pi}{\beta}$ -periodicity of \mathcal{H}_l ; and the bijectivity of σ allowing adequate index redefinitions.

The number of σ -stable indices is a priori arbitrary. However if there are more than two such indices, one will unavoidably have exact equality mod. T of at least two

position variables q and the invariant submanifold will actually lie in the singularity hyperplanes $q_i = q_j(T)$. This case must therefore be eliminated from our discussion, which leaves us with only three possibilities.

Case 1: For n even (odd number of sites), one may only consider the case of one stable index. This construction gives us the $BC_{\frac{n}{2}}$ RS model [11, 12, 10].

Case 2: For n odd (even number of sites), one may first consider the case of zero stable index. This leads to the $C_{\frac{n+1}{2}}$ case [11, 13].

Case 3: For n odd (even number of sites), one may then consider the case of two stable indices hereafter denoted 0 and $\bar{0}$. The only non-singular choice for values of the two fixed position coordinates is then $q_0 = 0$ and $q_{\bar{0}} = T/2$ up to trivial permutation. Connection to the superalgebra $C_{N+1}^{(2)}$ with $n \equiv 2N + 1$ will be established presently.

Remark 1: Regarding the rapidities, we will here restrict ourselves for the sake of simplicity to the choice $\theta_0 = \theta_{\bar{0}} = 0(\frac{2i\pi}{\beta})$. This implies immediately that the restriction of \mathcal{H}_0 to (2) is identically 1. We shall comment at the end (Comment 3, Section 5) on the implications of other possible choices.

We now establish the important properties of this consistent restriction. We first introduce a better adapted set of conjugate variables (for the full phase space) defined to be $\{\frac{1}{\sqrt{2}}(q_i - q_{\sigma(i)}), \frac{1}{\sqrt{2}}(\theta_i - \theta_{\sigma(i)})\}$; $\{\frac{1}{\sqrt{2}}(q_i + q_{\sigma(i)}) \equiv q_i^\perp, \frac{1}{\sqrt{2}}(\theta_i + \theta_{\sigma(i)}) \equiv \theta_i^\perp\}$, $\forall i, \sigma(i) \neq i$; finally $q_0 \equiv q_0^\perp, \theta_0 \equiv \theta_0^\perp, q_{\bar{0}} - \frac{T}{2} \equiv q_{\bar{0}}^\perp, \theta_{\bar{0}} \equiv \theta_{\bar{0}}^\perp$.

Corollary 1a: *These submanifolds are obviously endowed with a symplectic form $\{\}_{\text{rest}}$ where conjugate variables are respectively the restrictions of $\frac{1}{\sqrt{2}}(q_i - q_{\sigma(i)})$ and $\frac{1}{\sqrt{2}}(\theta_i - \theta_{\sigma(i)})$ for i such that $\sigma(i) \neq i$.*

Corollary 1b: *The Hamiltonians \mathcal{H}_l , restricted to the submanifolds (2) endowed with the symplectic form $\{\}_{\text{rest}}$ build an integrable hierarchy.*

Proof: An obvious rewriting of Proposition 1 states that the Hamiltonians \mathcal{H}_l expanded around the values $\theta_i^\perp = 0, q_i^\perp = 0$, have no linear term in $\theta_i^\perp, q_i^\perp$: indeed such terms would trigger a non-trivial dynamics of $\theta_i^\perp, q_i^\perp$ on (2). This leads us to define $\mathcal{H}_l^{\text{rest}} \equiv \mathcal{H}_l(\{\frac{1}{\sqrt{2}}(q_i - q_{\sigma(i)}), \frac{1}{\sqrt{2}}(\theta_i - \theta_{\sigma(i)}), q_i^\perp = 0, \theta_i^\perp = 0\})$ and now $\mathcal{H}_l = \mathcal{H}_l^{\text{rest}} +$ quadratic terms in $q_i^\perp, \theta_i^\perp$

In addition the \mathcal{H}_l Poisson-commute by construction. This is now expressed as :

$$\{\mathcal{H}_a, \mathcal{H}_b\}_{\text{full}} = 0 \equiv \{\mathcal{H}_a^{\text{rest}}, H_b^{\text{rest}}\}_{\text{full}} + \theta_i^\perp \times \cdots + q_i^\perp \times \cdots$$

which implies in particular: $\{\mathcal{H}_a^{\text{rest}}, H_b^{\text{rest}}\}_{\text{full}} = 0$.

Finally one has $\{\mathcal{H}_a^{\text{rest}}, H_b^{\text{rest}}\}_{\text{full}} \equiv \{\mathcal{H}_a^{\text{rest}}, H_b^{\text{rest}}\}_{\text{rest}} = 0$ by definition of the restricted symplectic structure on the submanifolds, thereby ending the proof.

For the sake of simplicity we shall from now on drop the “rest” index in \mathcal{H}_l .

Remark 2: In the elliptic case the potential function is biperiodic, hence one may allow for 4 fixed coordinates since two independent periods are available to define the invariant subspaces (replacing the single period congruence parameter T in (2)).

We now formulate:

Proposition 2 *The Hamiltonians K_l and K_{-l} are identical, and therefore equal to \mathcal{H}_l , on the submanifolds (2).*

The proof is a trivial consequence of the bijectivity of σ ; the fact that on the reduced submanifolds (2) one has up to a full period $\theta_{\sigma(i)} = -\theta_i$ and $q_{\sigma(i)} = -q_i$ for *all* indices; and the parity and periodicity properties of the potential f .

The immediate crucial consequence is that the restriction of the Hamiltonians \mathcal{H}_l can be rewritten as: $\mathcal{H}_l = P_l(\text{Tr} L^m)$ where P_l is the l -th Newton polynomial (sum of rank l minors) and L is the restriction of the Lax matrix of A_{2N+1} RS model to the submanifolds (2). We are therefore provided with a Lax representation for the integrable hierarchy \mathcal{H}_l .

We now discuss the particular dependance of the potential terms in the Hamiltonians, in order to justify the claimed connection to the twisted affine superalgebra $C_{N+1}^{(2)}$. Specific connection is the following: the fixing of q_0 to the value $T/2$ introduces an imaginary shift in the lattice describing the position dependance of the potential function, turning it into the root lattice of $C_{N+1}^{(2)}$.

More precisely the original dependance of the potential function was in the variables $q_i - q_j, i, j \in \{1 \cdots 2N + 2\}$, associating it with the root lattice of A_{2N+1} . After

this particular folding we now get a dependence in $q_i - q_j; q_i + q_j; q_i; 2q_i; q_i + T/2; i, j \in \{1 \cdots N\}$. This immediately leads to seek for a lattice with three root lengths together with the simultaneous existence of shifted and unshifted shortest roots. By exploration [14] the only possibility is $C_{N+1}^{(2)}$.

These variables may now be reexpressed as scalar products $\langle \alpha, Q \rangle$ where Q is the $(N + 1)$ -component vector in the dual of the Cartan algebra with coordinates $q_1 \cdots q_N$ on the dual of the Cartan algebra of the simple Lie algebra and T along the direction of the derivation generator d of the full Cartan algebra; α is any root of $C_{N+1}^{(2)}$ [14], exhibiting in particular the shift of the shortest roots by $\frac{1}{2}$ the derivation generator characteristic of $C_{N+1}^{(2)}$. It remains at this time still a formal connection, and we have no interpretation of the occurrence of a twisted affine superalgebra in this context.

The pure BC_N folding (case 1) differs qualitatively in that the one-body part of the potential for the first Hamiltonian (i.e. linearly dependent upon single exponentials of rapidities) does not contain a double pole at half period due to the absence of the extra dependence in $q_i + T/2$. In the non-relativistic CM limit however, the one-body potential in the first (quadratic) Hamiltonian obtained by *both* foldings exhibits double poles at half-period (occurring from the folding of terms $\sinh(q_i - q_j)^{-2}$ at $q_j = -q_i$ giving in particular a term $\cosh(q_i)^{-2}$ by doubling of the \sinh ; or at $q_j = \frac{T}{2}$); and double poles at integer period (occurring from the folding of terms $\sinh(q_i - q_j)^{-2}$ at $q_j = 0$). The technical reason is the multiplicative nature of the potential terms in a RS model as opposed to the additive nature thereof in a CM model, and is thus related to the fundamental difference between RS models realized [16] on a Heisenberg double [17, 18, 19] and CM models realized on a cotangent bundle [20]. Actually, the only difference between BC_N and $C_{N+1}^{(2)}$ foldings of CM models lies in the value of the residues (coupling constants) which in any case may be extended to take any value, hence it turns out to be irrelevant.

3 The $C_{N+1}^{(2)}$ Lax operator and r -matrix

We shall from now on use the notation $-i$ instead of $\sigma(i)$, defining that $-\bar{0} = \bar{0}$. The Lax formulation of $C_{N+1}^{(2)}$ RS system is therefore obtained as a folding of the A_{2N+1} case [11] explicitly built as follow:

We first label the $2N+2$ rapidities $\{\theta_i, i = -N \cdots 0, \bar{0}, \cdots N\}$ and conjugate positions $\{q_i, i = -N \cdots 0, \bar{0}, \cdots N\}$. Independent phase space variables on the restricted manifolds are here chosen to be $q_i, \theta_i, i \in \{1 \cdots N\}$. Note that on the restricted manifold the Poisson structure $\{\}_{rest}$ in terms of these variables reads $\{\theta_i, q_j\} = \frac{1}{2}\delta_{i,j}$ which will be responsible for an overall $\frac{1}{2}$ factor in the Poisson structure. This factor will be omitted from now on, corresponding to a normalization of the Poisson bracket as $\{\theta_i, q_j\} = \delta_{i,j}$. One thus identifies $\theta_i = \varepsilon_i \theta_{|i|}$ and $q_i = \varepsilon_i q_{|i|} + \delta_{i,\bar{0}} \frac{T}{2}$ with $\forall i \in \{1 \cdots N\}$, $\varepsilon_i = 1 = -\varepsilon_{-i}$ and $\varepsilon_0 = \varepsilon_{\bar{0}} = 0$.

The Lax matrix for the A_{2N+1} cases reads: $\mathcal{L} = \sum_{i,j=-N}^N \mathcal{L}_{ij} e_{ij}$ where $\mathcal{L}_{ij}(q_{-N}, \dots, q_N, \theta_j) = c(q_i - q_j) e^{-\beta \theta_j} f_{\{j\}}$, $\{e_{ij}\}$ is the standard basis for $(2N+2) \times (2N+2)$ matrices and $c(q) = \frac{\sinh \gamma}{\sinh(\frac{\gamma}{2} + \gamma)} = (1 - \lambda) \frac{z^{1/2}}{z - \lambda}$.

The Lax matrix for the $C_{N+1}^{(2)}$ Ruijsenaars-Schneider systems then reads:

$$L = \sum_{i,j=-N}^N L_{ij} e_{ij} \quad \text{with} \quad L_{ij} = \mathcal{L}_{ij}(-q_N, \dots, -q_1, 0, \frac{T}{2}, q_1, \dots, q_N, \varepsilon_j \theta_{|j|}) \quad (4)$$

It is now well-known that the Lax operator \mathcal{L} satisfies the quadratic fundamental Poisson bracket [21]: $\{\mathcal{L} \otimes \mathcal{L}\} = \mathcal{L} \otimes \mathcal{L} a_1 - a_2 \mathcal{L} \otimes \mathcal{L} + \mathcal{L}_2 s_1 \mathcal{L}_1 - \mathcal{L}_1 s_2 \mathcal{L}_2$, where $\mathcal{L}_1 = \mathcal{L} \otimes 1$, $\mathcal{L}_2 = 1 \otimes \mathcal{L}$ and the quadratic structure coefficients read: $a_1 = a + w$, $s_1 = s - w$, $a_2 = a + s - s^\pi - w$, $s_2 = s^\pi + w$. As usual, for any matrix $M = \sum_{ijkl=-N}^N M_{ijkl} e_{ij} \otimes e_{kl}$ the operation π is defined by: $M_{ijkl}^\pi = M_{klij}$. Matrices a, s, w take the form :

$$a = -\alpha \sum_{j,k=-N}^N a_{jk} e_{jk} \otimes e_{kj}, \quad s = \alpha \sum_{j,k=-N}^N s_{jk} e_{jk} \otimes e_{kk}, \quad w = \alpha \sum_{j,k=-N}^N a_{jk} e_{jj} \otimes e_{kk} \quad \text{with} \quad \alpha = \beta \frac{\nu}{2}$$

$$\text{and } a_{jk} = (1 - \delta_{j,k}) \coth \frac{\nu}{2} (q_j - q_k) = (1 - \delta_{j,k}) \frac{z_j + z_k}{z_j - z_k}, \quad s_{jk} = \frac{(1 - \delta_{j,k})}{\sinh \frac{\nu}{2} (q_j - q_k)}.$$

Remember that the most general structure of Poisson bracket for a Lax operator of a Liouville-integrable system is a linear one [22]: $\{L \otimes L\} = [r, L_1] - [r^\pi, L_2]$. The quadratic form corresponds to the general case [23] where the r -matrix itself assumes a linear dependence in L of form: $r = b L_2 + L_2 c$ with b and c arbitrary matrices yielding : $a_1 = c^\pi - c$, $a_2 = b^\pi - b$, $s_1 = c + b^\pi$ and $s_2 = s_1^\pi$.

We now straightforwardly extend the previous computation [10] of the BC_N Ruijsenaars-Schneider r -matrix. The Lax operator also satisfies a quadratic fundamental Poisson bracket, again exhibiting the dependence of the structure matrices a and s on both sets of dynamical variables. The occurrence of two zero-type indices 0 and $\bar{0}$ requires extra caution at one particular place, eventually leading to supplementary signs in the formula for factors of the r -matrix. Note finally that this derivation is equivalently applicable to the C_N -type RS Lax matrix, this time by altogether removing the zero-indices.

The r -matrix structure is again completely defined by a quadratic Poisson bracket with a_1 , a_2 , s_1 and s_2 changed into: $a_1 \rightarrow \tilde{a}_1 = a_1$, $a_2 \rightarrow \tilde{a}_2 = \tilde{a}_1 + \tilde{s}_1 - \tilde{s}_2 = a_2 + \tau^\pi - \tau$, $s_1 \rightarrow \tilde{s}_1 = s_1 + \sigma + \tau^\pi$, $s_2 \rightarrow \tilde{s}_2 = \tilde{s}_1^\pi = s_2 + \sigma^\pi + \tau = s_2 + \sigma + \tau$, with:

$$\tau = \alpha \sum_{i,k,l=-N}^N L_{k-i} L_{-il}^{-1} \left(\frac{u_{-i}}{2} - t_{k-i} \right) e_{ii} \otimes e_{kl}, \quad \sigma = \alpha \sum_{i,j=-N}^N (\delta_{i,j} s_i - (1 - \delta_{i,j}) a_{ij}) \frac{\mathcal{A}_j}{\mathcal{A}_i} e_{ij} \otimes e_{-j-i},$$

defining $t_{ij} = \coth\left(\frac{\nu}{2}(q_i - q_j) + \gamma\right) = \frac{z_i + \lambda z_j}{z_i - \lambda z_j}$, $s_i = \frac{1 + \lambda}{1 - \lambda} + \sum_{m=-N}^N \frac{t_{mi} + t_{im}}{2}$, $u_i =$

$$\sum_{k=-N}^N 2a_{ik} + t_{ki} - t_{ik} \text{ and } \mathcal{A}_i = \prod_{\substack{k=-N, \\ k \neq i}}^N \left(\frac{z_i - \lambda z_k}{\lambda z_i - z_k} \right)^{1/2} e^{-\beta \theta_i} e^{i\pi \delta_{i,\bar{0}}}.$$

Occurrence of the final sign factor $e^{i\pi \delta_{i,\bar{0}}}$ is the only qualitative modification induced by the supplementary fixed index $\bar{0}$. In addition the full r -matrix gets an overall $\frac{1}{2}$ factor due to the normalization of the restricted Poisson bracket.

As in the BC_N case this quadratic r -matrix structure is fully dynamical, depending both on the positions q_i 's and rapidities θ_i 's.

4 Koornwinder-van Diejen versus $C_{N+1}^{(2)}$ Ruijsenaars-Schneider Hamiltonians

As in the BC_N case the Hamiltonians \mathcal{H}_l generated by traces of powers of the $C_{N+1}^{(2)}$ Lax matrix (4) can be reexpressed in a more interesting form as:

$$\begin{aligned} \mathcal{H}_l &= \sum_{\substack{J \subset \{1..N\}, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} \mathcal{U}_{J^c, l-|J|} e^{-\beta \theta_{\varepsilon J}} f_{\varepsilon J}, \text{ with } \varepsilon J \equiv \{\varepsilon_j | j|, j \in J\} \text{ and} \\ \mathcal{U}_{K,p} &= \sum_{\substack{S \subset \mathcal{A}_K = K \cup -K \cup \{0, \bar{0}\} \\ S = -S, |S| = p}} \prod_{\substack{s \in S \\ k \in \mathcal{A}_K \setminus S}} f^{1/2}(q_s - q_k) = \sum_{\substack{S \subset \mathcal{A}_K \\ S = -S, |S| = p}} \prod_{\substack{s \in S \\ k \in \mathcal{A}_K \setminus S}} v(q_s - q_k). \end{aligned} \quad (5)$$

We now recall the form of classical Koornwinder-van Diejen Hamiltonians [5]:

$$H_l = \sum_{\substack{J \subset \{1..N\}, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} U_{J^c, l-|J|} e^{-\beta \theta_{\varepsilon J}} V_{\varepsilon J; J^c}^{1/2} V_{-\varepsilon J; J^c}^{1/2}, \quad (6)$$

where, after some rearrangements required to reintroduce the indices $0, \bar{0}$:

$$\begin{aligned} V_{\varepsilon J; K} &= \prod_{j \in \varepsilon J} w_r(q_j) \prod_{\substack{j \in \varepsilon J \\ k \in \mathcal{A}_K \cup -\varepsilon J}} v(q_j - q_k), \text{ with } w_r(q_j) = \frac{w(q_j)}{v(2q_j) v(q_j) v(q_j - \frac{T}{2})} \\ \text{and } U_{K,p} &= (-1)^p \sum_{\substack{\varepsilon I \subset \mathcal{A}_K \\ |I| = p}} \prod_{i \in \varepsilon I} w_r(q_i) \prod_{\substack{i, i' \in \varepsilon I \\ i < i'}} \frac{v(-q_i - q_{i'})}{v(q_i + q_{i'})} \prod_{\substack{i \in \varepsilon I \\ k \in \mathcal{A}_K \setminus \varepsilon I}} v(q_i - q_k). \end{aligned} \quad (7)$$

The potentials w are particular functions explicited in [3] and may be interpreted physically as an interaction with some external field.

Direct computation yields: $V_{\varepsilon J; J^c} V_{-\varepsilon J; J^c} = \prod_{j \in \varepsilon J} w_r(q_j) w_r(-q_j) f_{\varepsilon J}^2$.

Setting $w_r(q_j) = 1$, that is $w(q_j) = v(2q_j) v(q_j) v(q_j - \frac{T}{2})$, which is an admissible choice according to [5], H_l (6) takes actually the same form as \mathcal{H}_l (5), up to the crucial change of $\mathcal{U}_{K,p}$ into $U_{K,p}$. They are generally not equal, except for $p = 0$,

where trivially: $U_{K,0} = 1 = \mathcal{U}_{K,0}$. When $p = 1$, one gets:

$$U_{K,1} = - \sum_{i \in \mathcal{A}_K \setminus \{0, \bar{0}\}} \prod_{\substack{k \in \mathcal{A}_K \\ k \neq i}} v(q_i - q_k) \quad \text{and} \quad \mathcal{U}_{K,1} = \prod_{\substack{k \in \mathcal{A}_K \\ k \neq 0}} v(q_k) + \prod_{\substack{k \in \mathcal{A}_K \\ k \neq \bar{0}}} v\left(\frac{T}{2} - q_k\right).$$

Evaluation of a suitable contour integral as in [10] gives the Liouville-type functional identity: $\sum_{i \in \mathcal{A}_K} \prod_{\substack{k \in \mathcal{A}_K \\ k \neq i}} v(q_i - q_k) = \frac{\sinh \gamma (2|K|+2)}{\sinh \gamma}$, that is: $U_{K,1} = \mathcal{U}_{K,1} - \frac{\sinh \gamma (2|K|+2)}{\sinh \gamma}$.

We now recall the general theorem established in [10]:

Theorem 1 *Let q_i and θ_i , $i \in \mathbb{N}$, be a set of conjugated variables such that $\{\theta_i, q_j\} = \delta_{ij}$. Let I and K be arbitrary finite sets of indices included in \mathbb{N} . Assume the existence of a set of complex functions $u_{K,p}$ depending upon the set of indices K and a natural integer p , and of another set of complex functions $v_{\varepsilon J, I}$ depending upon the sets of indices J and I ($J \subset I$) and a $|J|$ -uple of signs $\varepsilon = (\varepsilon_j, j \in J)$, such that:*

- $u_{K,p}$ and $v_{\varepsilon J, I}$ be independent of the rapidities θ_i s.
- $u_{K,0} = 1$, $v_{\emptyset, I} = 1$, and $v_{\varepsilon\{j\}, I} \neq 0$.
- $S^I = \{ h_l^I = \sum_{\substack{J \subset I, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} u_{J^c, l-|J|} e^{-\beta \theta_{\varepsilon J}} v_{\varepsilon J, I}, l \in \{1..|I|\} \}$ be a family of Poisson-

commuting functions ($\theta_{\varepsilon J} = \sum_{j \in J} \varepsilon_j \theta_j$).

If there exists a second set of complex functions $\tilde{u}_{K,p}$ obeying the first two conditions; such that $\tilde{S}^I = \{ \tilde{h}_l^I = \sum_{\substack{J \subset I, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} \tilde{u}_{J^c, l-|J|} e^{-\beta \theta_{\varepsilon J}} v_{\varepsilon J, I}, l \in \{1..|I|\} \}$ be a new family of Poisson-commuting functions; and $\tilde{u}_{K,1} = u_{K,1} + c_1(|K|)$, then there exist coefficients $c_r(m)$, $(r, m) \in \mathbb{N}^2$, independent of all dynamical variables, connecting the two families of Hamiltonians as: $\tilde{h}_l^I = \sum_{s=0}^l c_{l-s}(|I|-s) h_s^I$, with $\forall m \in \mathbb{N}, c_0(m) = 1$.

Hence the two relations for $p = 0$ and $p = 1$ between the $U_{K,p}$'s and $\mathcal{U}_{K,p}$'s are sufficient to establish that the two sets of Hamiltonians define the same family of commuting dynamical flows, namely one set of Hamiltonians is a triangular linear combination of the other set.

5 Comments

1. It follows from the uniqueness theorem that the $C_{N+1}^{(2)}$ RS hierarchy defined here is equivalent to the KvD hierarchy for a particular set of couplings $\mu, \mu, \frac{\mu}{2}, \frac{\mu}{2}$. However as already emphasized the full KvD pole structure is now obtained, contrary to the pure BC_N case where one coupling constant is actually set to zero. To obtain the complete KvD set of hyperbolic potentials one clearly needs to define an “extension” (with the same meaning as the “extension” of BC_N CM models in [9, 8] leading to the full Inozemtsev potentials) of the $C_{N+1}^{(2)}$ Lax formulation. We hope to report on this problem soon.

2. In the elliptic case where 4 coordinates q are fixed to various combinations of the two half-periods, the folding leads to a one-body potential with 4 double poles located at q_i — (half-integer linear combinations of the two periods): indeed the term depending on $2q_i$ leads to a duplication of the 4 poles induced by the 4 fixed coordinates.

This is the correct denumbering and location of poles in the classical limit of the quantum general elliptic Hamiltonian constructed in [7], where 4 pairs of poles separated by an order \hbar degenerate to these 4 double poles. Of course, here again one does not obtain the full set of coupling constants (or equivalently the residues at the poles). The conjecture that folded RS models are the correct starting points for construction of a full KvD-Hikami-Komori classical Lax-type representation may therefore be extended to the elliptic case, with a suitably generalized underlying root lattice. More precisely, we conjecture that this new root lattice is associated with twisted toroidal superalgebras. Indeed it exhibits three root lengths (pointing to superalgebras) with half-period shifts (pointing to twisted algebras), precisely two independent shifts of the shortest roots, interpreted as corresponding to the two independent derivations characteristic of toroidal algebras.

3. We have restricted ourselves here to the choice $\theta_0 = \theta_{\bar{0}} = 0$. In fact one can also choose either or both to be equal to a half-period $\frac{i\pi}{\beta}$. The corresponding phase space manifold is also invariant under the Hamiltonians \mathcal{H}_l . These Hamiltonians therefore

define an integrable hierarchy. At first sight it is not identical to the one defined here: the Hamiltonians indeed exhibit modifications of several relative signs in the potential terms; for instance the pure potential in the first Hamiltonian will become $\mathcal{U}_{K,1} =$

$$\prod_{\substack{k \in \mathcal{A}_K \\ k \neq 0}} v(q_k) - \prod_{\substack{k \in \mathcal{A}_K \\ k \neq \bar{0}}} v\left(\frac{T}{2} - q_k\right) \text{ if } \theta_{\bar{0}} = \frac{i\pi}{\beta}.$$

Whether these hierarchies are genuinely *new* integrable systems or may be obtained from our original $C_{N+1}^{(2)}$ Hamiltonians by some redefinition of parameters is an open question which we will not address here.

Acknowledgements

We wish to thank Luc Frappat for his help in unraveling the meaning of the underlying algebraic structure of the folding.

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